Ergodic Theory
Problem Sheet 5

Course Instructor: Florian K. Richter
Problems by: Jovan Andreevski, Dimitrios Charamaras

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P1. Show that every irrational number has a continued fraction expansion and that this representation is unique.

P2. Let $u = [0; a_1, a_2, \ldots] \in (0,1)$, and let $\frac{p_n}{q_n}$ be its n-th convergent. Show that for any $n \in \mathbb{N}$

$$\left| u - \frac{p_n}{q_n} \right| > \frac{1}{q_n q_{n+2}}.$$

P3. A real number $u = [0; a_1, a_2, \ldots] \in (0,1)$ is called *badly approximable* if there exists some $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \geq 1$.

Show that $u \in (0,1)$ is badly approximable if and only if there exists some $\varepsilon > 0$ such that

$$\left| u - \frac{p}{q} \right| \geq \frac{\varepsilon}{q^2}$$

for all rational numbers $\frac{p}{q}$.

P4. Let $(X, \mathscr{B}, \mu, T)$ be an ergodic measure-preserving system and let A be a set of positive measure. The pointwise ergodic theorem shows that for almost all $x \in X$, the set of visiting times

$$\Lambda_x = \{n \in \mathbb{N} \colon T^n x \in A\}$$

has natural density equal to $\mu(A)$. Is it true that, for almost all $x \in X$, the set Λ_x has bounded gaps, i.e. it is *syndetic*?